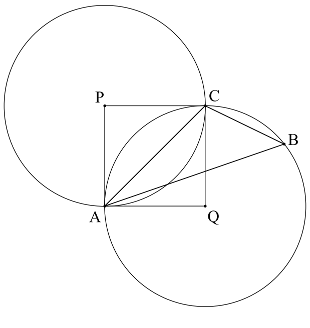
University of Mississippi

3^{rd} Annual High School Mathematics Contest
Team Competition Solutions
October 27, 2007

1. Let A and C be points in the plane with $AC = 1$. Find the area of the set of all points B so that $\angle ABC \geq 45°$.

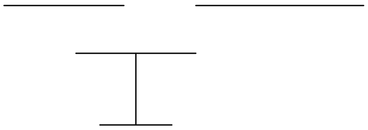


Solution. The set in question consists of the union of two disks with centers P and Q such that $m\angle APC = m\angle AQC = 90°$. This can be seen because $m\angle ABC = \frac{1}{2}m\overset{\frown}{\mathrm{AC}} = \frac{1}{2}(90°) = 45°$. If B is inside these two circles on the same side of $\overleftrightarrow{AC}$ as Q, then let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$ intersect circle Q at the points D and E (different from A and C). Then we have $m\angle ABC = (m\overset{\frown}{\mathrm{AC}} + m\overset{\frown}{\mathrm{DE}})/2 > m\overset{\frown}{\mathrm{AC}}/2 = 45°$. Also, if B is outside the circle, let D and E be defined similarly. Then $m\angle ABC = (m\overset{\frown}{\mathrm{AC}} - m\overset{\frown}{\mathrm{DE}})/2 < m\overset{\frown}{\mathrm{AC}}/2$. Thus the region we have specified is the one in question. Now its area is twice the sum of the area of $\triangle AQC$ and the sector AQC. This is $2\left[\frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{3}{4}\pi\left(\frac{1}{\sqrt{2}}\right)^2\right] = \boxed{1/2 + 3\pi/4}$.

2. Fermat's little theorem states that whenever p is prime and a is not a multiple of p, a^{p-1} leaves a remainder of 1 when divided by p. Given that 21^5 does not leave a remainder of 1 when divided by 11, what is the remainder when 21^5 is divided by 11?

Solution. Using Fermat's little theorem, we obtain the result that $21^{10} - 1$ is divisible by 11. Writing $21^{10} - 1 = (21^5 + 1)(21^5 - 1)$ and using the fact that $21^5 - 1$ is not divisible by 11, we know that $21^5 + 1$ is divisible by 11, which means that 21^5 leaves a remainder of $\boxed{10}$ when divided by 11.

3. A certain rectangular doorway has dimensions $7' \times 3'$. A table has a square top that is $5' \times 5'$, and it is connected via a $3\frac{1}{2}'$ pole protruding from the center of the square to a circular base that is $3'$ in diameter. A picture of the doorway and table are shown to the right with the table turned on its side. Determine whether it is possible to pass the table through the doorway, and either explain how to do it or prove that it is impossible. Assume that the walls, the table top, the pole, and the base are all negligibly thin.



Solution. The table can be passed through the doorway. Turn the table slightly with respect to the doorway and pass half the table top through, rotate the table to get the other half through, and then apply the same technique to the base. According to this strategy, the sufficient conditions on the doorway are that both its dimensions exceed half the side length of the tabletop as well as the radius of the base, and that one of its dimensions exceed the side length of the base. These conditions are satisfied since $3 > 2.5' > 1.5'$, and $7' > 5'$.

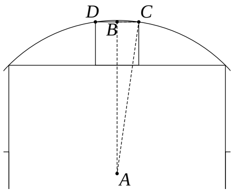
4. Given that $x_1, x_2, x_3, \ldots, x_{100}$ are all integers and that

$$x_1 x_2 x_3 \cdots x_{99} x_{100} = -1,$$

how many different possibilities are there for the ordered 100-tuple $(x_1, x_2, \ldots x_{100})$? Express your answer as p^n, where p is a prime number and n is a positive integer.

Solution. First, observe that all the variables must be ± 1. Then for each $i = 1, 2, 3 \ldots, 99$, we may take $x_i = +1$ or $x_i = -1$ freely, and then x_{100} has to be equal to the product $x_1 x_2 \cdots x_{99}$. Thus we have 99 decisions to make with two choices for each of them, so the total number of ways to make these decisions is $\boxed{2^{99}}$.

5. A square is inscribed in a circle, and four smaller squares are constructed that have two vertices on a side of the larger square and the other two vertices on the circle. Find the area of one of these smaller squares.



Solution. Let point A be the center of the circle, and let points B and C be the midpoint and a vertex respectively of the distant side of one of the small squares. Then $\triangle ABC$ is a right triangle. To prove this, let D be the vertex shown in the diagram, and note that $\triangle ABC \cong \triangle ABD$ because they have all three pairs of sides congruent. Therefore $\angle ABD \cong \angle ABC$. Since $m\angle ABD + m\angle ABC = 180°$, we have $m\angle ABC = 90°$. Now the side length S of the large square satisfies $S\sqrt{2} = 2$, so $S = \sqrt{2}$. Then, using $\triangle ABC$, the small square side length s satisfies $(s + S/2)^2 + (s/2)^2 = 1^2$, so $s = \boxed{\sqrt{2}/5}$.

6. Five very small marks are made on the surface of a solid, opaque cube. Prove that no matter how the marks are made, there must be a perspective from which one may view the cube so that at least three of the marks are visible.

Solution. Choose one of the vertices of the cube, and view the cube with this vertex nearest you. By viewing the cube similarly from the opposite vertex, we have two different perspectives one of which enables three faces to be seen and the other of which enables the other three faces to be seen. If each of these sets of faces has no more than two marks on it, then there can be no more than 4 marks on the cube, a contradiction. Hence one of the perspectives enables us to see at least three marks.

7. Find all real solutions of the equation

$$(3x - 63)(x - 47) = x^5 - 21x^4 + 13x - 273.$$

Solution. Factor and rearrange:

$$\begin{aligned}(3x-63)(x-47) &= x^5 - 21x^4 + 13x - 273.\\ 3(x-21)(x-47) &= x^4(x-21) + 13(x-21).\\ 3(x-21)(x-47) - x^4(x-21) - 13(x-21) &= 0.\\ (x-21)(3x - 141 - x^4 - 13) &= 0.\\ (x-21)[3x - (x^4 + 154)] &= 0.\end{aligned}$$

Now we have either $x = 21$ or $3x = x^4 + 154$. The latter case is impossible, since we have $x^4 + 154 > 3x$ for all x. For $x < 0$, this may be seen from the fact that the left-hand side is positive whereas the right-hand side is negative. For $0 \le x \le 3$, we can see that $154 > 3x$, and for $3 < x$, we have $(x - 3) > 0 \implies x^3(x - 3) > 0 \implies x^4 > 3x$. Therefore, the only real solution is $\boxed{x = 21}$.

8. Forty-five distinct playing cards are placed in a rectangular array of five columns and nine rows. An observer chooses a card secretly and tells the dealer which of the five columns the card is in. The dealer then puts the cards into a stack, picking up one column at a time. The column specified by the observer is picked up third, and the cards from each column are picked up in order from top to bottom. The cards are again dealt in the rectangular array, the first row being dealt in order from left to right, then the second row, and so on. The observer again specifies the column that contains his card, and the dealer again picks up the columns one at a time, the column specified being picked up third. This process is repeated one more time. Prove that the observer's card will be the twenty-third card in the deck.

Solution. When the cards are first picked up, the card's location is somewhere between 19th and 26th card in the deck, because its column is picked up third. When the cards are dealt the second time, then, the card may be no higher than the fourth row (since $5 \times 3 = 15 < 19$) and no lower than the the sixth row (since $5 \times 6 = 30 > 26$). Therefore, when the cards are picked up the second time, the card's location in the deck must be between 18+4=22nd and 18+6=24th. When the cards are dealt a final time, then, the card must be in the fifth row (since $5 \times 4 = 20 < 22$ and $5 \times 5 = 25 > 24$). Therefore when the cards are picked up for the last time, the card must be the $18 + 5 = 23$rd card.

9. In terms of n, find a formula for the determinant of an $n \times n$ matrix whose diagonal entries are all 0 and the rest of whose entries are 1.

<u>Solution.</u> For $i = 2, 3, \ldots, n$, we replace the ith row with the sum of the ith row and the first row. Then for $j = 2, 3, \ldots, n$, we replace the first row with the sum of the first row and the jth row. The determinant is then easily evaluated by expanding along the first row:

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{vmatrix} = \begin{vmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{vmatrix} = (n-1) \begin{vmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{vmatrix}$$

Now the determinant of a diagonal matrix is the product of the diagonal entries, so the determinant in question is $\boxed{(-1)^{n-1}(n-1)}$.

10. Prove that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} + \frac{1}{100} > \pi.$$

<u>Solution.</u> First we produce a lower bound for the sum

$$S(a,b) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{b}$$

where $a < b$ are integers and for the sake of simplicity $b - a$ is even so that the sum $S(a,b)$ has an odd number of summands. Set $m = (a+b)/2$, the median of the numbers $a, a+1, \ldots, b$. We group terms by adding first to last, second to next-to-last, and so on (written here from the middle out):

$$\begin{aligned} S(a,b) &= \frac{1}{m} + \left(\frac{1}{m+1} + \frac{1}{m-1}\right) + \left(\frac{1}{m+2} + \frac{1}{m-2}\right) + \cdots + \left(\frac{1}{m + \frac{b-a}{2}} + \frac{1}{m - \frac{b-a}{2}}\right) \\ &= \frac{1}{m} + \frac{2m}{m^2-1} + \frac{2m}{m^2-4} + \cdots + \frac{2m}{m^2 - \left(\frac{b-a}{2}\right)^2} \\ &> \frac{1}{m} + \underbrace{\frac{2m}{m^2} + \frac{2m}{m^2} + \cdots + \frac{2m}{m^2}}_{(b-a)/2 \text{ terms}} \\ &= \frac{1}{m} + \frac{2}{m} \cdot \frac{b-a}{2} \\ &= \frac{2(b-a+1)}{b+a} \end{aligned} \tag{1}$$

Now applying (1) to $S(2,100)$ gives $S(2,100) > 198/102$, which is not sufficient to prove that $S(2,100) > \pi$. However, we can improve our lower bound by adding the first few terms of our sum precisely and applying (1) to the rest. A little trial and error reveals that the best number of terms to add manually is 6, and a little long division is necessary to compare the result to π:

$$\begin{aligned} S(2,100) &= S(2,7) + S(8,100) \\ &> \frac{223}{140} + \frac{93}{54} \\ &> 1.59 + 1.72 \\ &= 3.31 \\ &> \pi, \end{aligned}$$

which is what we wanted to prove. Numerical computation reveals that the actual value of $S(2,100)$ is 4.187, to the nearest thousandth.