

University of Mississippi

2nd Annual High School Mathematics Contest

Team Competition Solutions

October 21, 2006

1. For real numbers r , let $\lfloor r \rfloor$ be the greatest integer that is less than or equal to r . Solve the inequality

$$\lfloor x \rfloor + \lfloor x + 3 \rfloor \leq 17$$

Solution. Since $\lfloor n + r \rfloor = n + \lfloor r \rfloor$ for integers n , we may write

$$\lfloor x \rfloor + \lfloor x + 3 \rfloor \leq 17$$

$$\lfloor x \rfloor + \lfloor x \rfloor + 3 \leq 17$$

$$2\lfloor x \rfloor \leq 14$$

$$\lfloor x \rfloor \leq 7,$$

from which we may see that $\boxed{x < 8}$.

2. Find the minimum distance from the point whose cartesian coordinates in space are $(8, 12, -9)$ to some point on a unit sphere centered at the origin.

Solution. Denote by P the point $(8, 12, -9)$, by O the origin, by Q the point on the sphere that minimizes the distance PQ . Using the fact that the shortest distance between two points is a straight line, we find that $PQ + OQ \geq OP$, and from this we conclude that $PQ \geq \sqrt{8^2 + 12^2 + (-9)^2} - 1 = \boxed{16}$. It takes on this value when Q 's coordinates are $(\frac{8}{17}, \frac{12}{17}, \frac{-9}{17})$.

3. Rationalize the denominator and simplify:

$$\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$$

Solution. Since there are three radicals on the bottom, we have to clear them in two steps:

$$\begin{aligned} \frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}} \cdot \frac{\sqrt{2} - (\sqrt{3} + \sqrt{5})}{\sqrt{2} - (\sqrt{3} + \sqrt{5})} &= \frac{\sqrt{2} - (\sqrt{3} + \sqrt{5})}{2 - (8 + 2\sqrt{15})} \\ &= \frac{\sqrt{5} + \sqrt{3} - \sqrt{2}}{6 + 2\sqrt{15}} \\ &= \frac{\sqrt{5} + \sqrt{3} - \sqrt{2}}{6 + 2\sqrt{15}} \cdot \frac{6 - 2\sqrt{15}}{6 - 2\sqrt{15}} \\ &= \boxed{\frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}} \end{aligned}$$

4. What is the maximum number of bishops that may be placed on an 8×8 chess board without any bishop being in a square threatened by any other bishop? (A bishop threatens any square that lies on the same diagonal as the bishop).

Solution. First we color the chess board with alternating black and white squares and notice that no bishop on a white square may threaten a bishop on a black square (and vice versa). Therefore, the problem reduces to maximizing the number of bishops that may be placed on the black squares. Since there are 7 diagonals that run in the direction of the main black diagonal, there may be no more than 7 total bishops on black squares. But we see that placing 7 bishops on black squares is possible by putting them at the ends of the board. Therefore, the maximum total number of bishops that may be placed on the board is $7 + 7 = \boxed{14}$.

5. A certain pattern is used to generate a sequence of numbers, the first few of which are shown below. Prove that no matter how far this sequence is continued, no 4 will ever appear as a digit.

1
 11
 21
 1211
 111221
 312211
 13112221
 1113213211
 ...

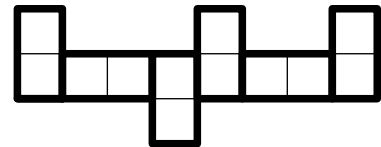
Solution. The pattern is best communicated by example: to get from the fourth line to the fifth line above, read the fourth line: "There are three ones, two twos, and one one." Now in order for a 4 to appear as a digit, some line must contain four consecutive copies of a certain digit, call that digit b . This block of four b 's will be straddled by two different digits, call them a and c . Therefore our string looks like $\cdots abbbbc \cdots$. This first two b 's may have appeared for only two reasons: in the previous line, there were either a b 's followed by b b 's, or there were some number of a 's preceding b b 's. The former case is impossible, because we would have written $(a + b)b$ instead of $abbb$. In the latter case, we have a similar contradiction, since the latter two b 's in the string of four indicate that we should have written $(b + b)b$ instead of $bbbb$.

6. If both a and b are strictly between 0 and 1, prove that $a + b - ab$ must be strictly between 0 and 1.

Solution 1. If a and b are probabilities of two events, the probability that at least one of those events will occur is $a + b - ab$ by the principle of inclusion-exclusion. Since there is some non-zero probability of both events not occurring, this number must be strictly between 0 and 1.

Solution 2. Write $a + b - ab = 1 - (a - 1)(b - 1)$. Since $(a - 1)(b - 1) = (1 - a)(1 - b)$ is the product of two numbers between 0 and 1, it also is between 0 and 1. Therefore, $1 - (a - 1)(b - 1)$ is also between 0 and 1.

7. Dominoes of dimension $2'' \times 1''$ are used to cover a $1'' \times 8''$ strip of $1'' \times 1''$ squares. Dominoes may be placed either along the strip so as to cover two of the squares, or they may be placed so that one square inch of the domino lies above or below the strip. One such covering for a strip for which is shown. Find the number of ways there are to cover the strip. (Coverings that differ by a rotation are considered distinct.)



Solution. Define t_n to be the number of ways to cover a $1'' \times n''$ strip of squares. Notice by inspection that $t_1=2$ and $t_2=5$. Notice also that to cover a strip of arbitrary length we must either include a horizontally positioned domino to cover the last two squares and cover the previous $n - 2$ tiles in t_{n-2} ways or we must cover the last square with a vertically positioned domino and cover the previous $n - 1$ dominoes in t_{n-1} ways. This leads us to the formula $t_n = 2t_{n-1} + t_{n-2}$, which may be used to find $t_3 = 12, t_4 = 29, \dots, t_8 = \boxed{985}$.

8. Evaluate

$$\frac{\sum_{k=0}^{20} \cos\left(\frac{\pi(k-5)}{20}\right)}{\sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)}$$


Solution. Start by splitting up the argument of cosine, then rearrange according to the cosine difference angle formula:

$$\frac{\sum_{k=0}^{20} \cos\left(\frac{\pi(k-5)}{20}\right)}{\sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)} = \frac{\sum_{k=0}^{20} \cos\left(\frac{\pi k}{20} - \frac{\pi}{4}\right)}{\sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)}$$

$$= \frac{\sum_{k=0}^{20} \frac{1}{\sqrt{2}} \cos\left(\frac{\pi k}{20}\right) + \sum_{k=0}^{20} \frac{1}{\sqrt{2}} \sin\left(\frac{\pi k}{20}\right)}{\sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)}$$

Now notice that the first sum in the numerator disappears since $\cos(\pi - \theta) = -\cos\theta$. Therefore, we have

$$\frac{\sum_{k=0}^{20} \frac{1}{\sqrt{2}} \cos\left(\frac{\pi k}{20}\right) + \sum_{k=0}^{20} \frac{1}{\sqrt{2}} \sin\left(\frac{\pi k}{20}\right)}{\sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)} = \frac{\frac{1}{\sqrt{2}} \sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)}{\sum_{k=0}^{20} \sin\left(\frac{\pi k}{20}\right)} = \boxed{\frac{1}{\sqrt{2}}}$$

9. A flat, triangular parking lot has dimensions 15 car lengths by 20 car lengths by 25 car lengths, where a “car length” is a fixed unit of measure that is a little longer than the average car’s length. A parking space must be rectangular, with dimensions one car length long by one-half car length wide, and every space must have 1.5 car lengths of clearance behind it (area within the parking lot where there is no parking space). Different parking spaces may share the same clearance space, and every spot must be accessible from some point on the perimeter of the lot, not counting the curb that bounds that parking space itself. For simplicity, assume that the long side of every space is perpendicular to the 15-car-length side of the lot. Determine the maximum number of parking spaces the lot can hold.
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Solution. Notice first that the shape of the lot is a right triangle. The row that will be able to hold the greatest number of cars is the one bordering the 20-car-length side. Since this row needs 2.5 car lengths of clearance, this row will be able to hold $\lfloor 20 - \frac{4}{3} \cdot 2.5 \rfloor = 16$ cars, where we have used the fact that the triangle in the corner cut off by the 2.5-car-length restriction is similar to the whole 15-20-25 triangle. Now, we may put a row of cars immediately opposite the 1.5 car lengths of space. Therefore, the next row will be able to hold $\lfloor 20 - \frac{4}{3} \cdot 3.5 \rfloor = 15$ cars. Then, for the next row of cars we again need 2.5 car lengths of space, so this row holds $\lfloor 20 - \frac{4}{3} \cdot 6 \rfloor = 12$ cars.

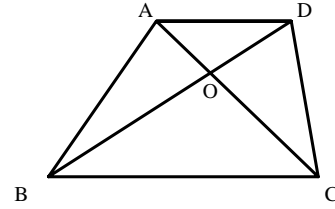
We continue in this manner to find a total of $\sum_{h \in H} \lfloor 20 - \frac{4}{3} \cdot h \rfloor = \boxed{69}$, where $H = \{2.5, 3.5, 6, 7, 9.5, 10.5, 13, 14\}$.

10. Let $[\mathcal{F}]$ represent the area of figure \mathcal{F} . If $ABCD$ is a trapezoid with $AD \parallel BC$ and O is the intersection of the trapezoid's diagonals, prove that

$$\frac{[AOB]}{[ABCD]} < \frac{1}{4}$$

Solution. Because $AD \parallel BC$, we have that $\triangle AOD \sim \triangle COB$. Also, from the general triangle area formula $\frac{1}{2}ab \sin C$, we have that

$$\begin{aligned} \frac{BD}{BO} &= \frac{[BCD]}{[BOC]} \\ \frac{BO+DO}{BO} &= \frac{[DOC]+[BOC]}{[BOC]} \\ \frac{DO}{BO} &= \frac{[DOC]}{[BOC]} \end{aligned}$$



Identical reasoning produces the result that $\frac{AO}{CO} = \frac{[AOB]}{[BOC]}$. But by the similarity $\triangle AOD \sim \triangle COB$, we find $AO/CO = OD/OB$, so that

$$[AOB] = [DOC]. \tag{1}$$

Now, let us call b_1 the smaller base, h_1 the perpendicular drawn from it to point O , b_2 the larger base and h_2 the perpendicular drawn from it to point O , and h the perpendicular distance from \overline{AD} to \overline{BC} . Notice that the similarity $\triangle AOD \sim \triangle COB$ implies that $h_1 < h_2$. Now we will show that $[AOD] + [BOC] > \frac{1}{2}[ABCD]$:

$$\begin{aligned} (b_2 - b_1)(h_2 - h_1) &> 0 \\ h_1 b_1 + h_2 b_2 &> h_2 b_1 + h_1 b_2 \\ 2h_1 b_1 + 2h_2 b_2 &> h_2 b_1 + h_1 b_2 + h_1 b_1 + h_2 b_2 \\ \frac{1}{2}(h_1 b_1 + h_2 b_2) &> \frac{1}{2} \cdot \frac{1}{2} \cdot (h_1 + h_2)(b_1 + b_2) \\ \frac{1}{2}h_1 b_1 + \frac{1}{2}h_2 b_2 &> \frac{1}{2} \cdot \frac{1}{2} \cdot (h)(b_1 + b_2) \\ [AOD] + [BOC] &> \frac{1}{2}[ABCD] \end{aligned} \tag{2}$$

Therefore, combining (1) and (2), we find

$$\begin{aligned} [AOB] &= \frac{1}{2}([ABCD] - [AOD] - [BOC]) \\ &< \frac{1}{2}([ABCD] - \frac{1}{2}[ABCD]) \\ &= \frac{1}{4}[ABCD], \end{aligned}$$

as desired.