1. If \( a + b + c = 1 \) with \( a, b, c \geq 0 \), show that
\[
\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{3}.
\]

**Solution:** We observe
\[
(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 = a + b + c + 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac}
\]
\[
\leq a + b + c + a + b + b + c + a + c
\]
\[
= 3(a + b + c) = 3.
\]
The desired inequality follows by taking square roots.

2. Compute the sum
\[
S_n = \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2)(3n+1)}.
\]

**Solution:** We notice that
\[
\frac{1}{3k-2} - \frac{1}{3k+1} = \frac{3}{(3k+1)(3k-2)}.
\]
Hence,
\[
3S_n = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \cdots + \frac{1}{3n-2} - \frac{1}{3n+1}
\]
\[
= 1 - \frac{1}{3n+1} = \frac{3n}{3n+1}.
\]
Hence,
\[
S_n = \frac{n}{3n+1}.
\]

3. Consider any three consecutive positive integers. Show that the cube of the largest cannot be the sum of the cubes of the other two.

**Solution:**
Suppose that \( a^3 = (a - 1)^3 + (a - 2)^3 \) for an integer \( a \geq 3 \).

Note first that \( a \) cannot be even, since then the LHS would be even and the RHS odd.

Now suppose that \( a \) is odd. We consider the expression modulo 4 (i.e. the integer remainder when we divide by 4). Since \( a \) is odd, \( a - 1 \) is even so \((a - 1)^3\) is divisible by 4. However, \( x^3 \equiv x \pmod{4} \) for odd \( x \), but \( a \) and \( a - 2 \) must have different remainders modulo 4. This also leads to a contradiction.

4. What is the largest number by which \( n^3 - n \) is divisible for all possible integers \( n \)?

**Solution:**
Observe that \( n^3 - n = n(n^2 - 1) = n(n-1)(n+1) \), and thus is a product of three consecutive integers. At least one of these three is a multiple of 3 and at least one is a multiple of 2, thus it is divisible by 6 for any \( n \).

Now note that for \( n = 2 \), we obtain \( 8 - 2 = 6 \), so 6 is the largest such number.
5. What is the last digit of
\[ 3^{2014} - 2^{2014} \]

**Solution:** It suffices to consider this expression modulo 10 (i.e. the remainder when we divide by 10). Note first that 2014 is even but not divisible by 4, so 2014 = 4k + 2 for some k.

We also notice that modulo 10, \( 2^2 = 4, 2^4 = 6 \) and \( 2^4 \cdot 2^4 = 6 \), and hence \( 2^{4k} \equiv 6 \pmod{10} \). Thus,

\[
2^{2014} \equiv 2^{4k} \cdot 2^2 \pmod{10} \\
\equiv 6 \cdot 4 \pmod{10} \\
\equiv 4 \pmod{10}
\]

Also observe that \( 3^2 \equiv 9 \pmod{10} \) and \( 3^4 \equiv 1 \pmod{10} \), so \( 3^{2014} \equiv 9 \pmod{10} \).

Thus, our expression is congruent to 5 modulo 10, and hence 5 is the last digit.

(This can also be done by considering the resulting pattern.)

6. Suppose that
\[
\sin \alpha + \sin \beta = m \\
\cos \alpha + \cos \beta = n \quad \text{with } m^2 + n^2 \neq 0.
\]

(a) Find \( \cos(\alpha - \beta) \).

(b) Find \( \sin(\alpha + \beta) \).

**Solution:**

First, we square both equations.

\[
m^2 = \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \\
n^2 = \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta.
\]

Adding, we obtain

\[
m^2 + n^2 = 2 + 2 \cos(\alpha - \beta).
\]

Hence, \( \cos(\alpha - \beta) = \frac{m^2 + n^2 - 2}{2} \).

Now, for the second half, we will use the observation

\[
\sin(\alpha + \beta) \cos(\alpha - \beta) = \sin \alpha \cos \alpha + \sin \beta \cos \beta.
\]

We compute the product of our two given formulas:

\[
mn = \sin \alpha \cos \alpha + \sin \alpha \cos \beta + \sin \beta \cos \alpha + \sin \beta \cos \beta \\
= \sin(\alpha + \beta) + \sin(\alpha + \beta) \cos(\alpha - \beta) \\
= \sin(\alpha + \beta) (1 + \cos(\alpha - \beta)) \\
= \sin(\alpha + \beta) \frac{m^2 + n^2}{2}.
\]

We then get

\[
\sin(\alpha + \beta) = \frac{2mn}{m^2 + n^2}.
\]
7. Let \( a, b \) and \( c \) be real numbers satisfying the inequalities:

\[
|a| \leq |b-c|, \quad |b| \leq |c-a|, \quad |c| \leq |a-b|.
\]

Show that one of these numbers is the difference of the other two.

**Solution:**

Observe first of all that if \( a, b, c \) satisfy these inequalities, so do \(-a, -b, -c\). Thus, we may assume that at least two of the three are non-negative.

Without loss of generality, also take that \( a \leq b \leq c \), and thus \( b, c \) are non-negative.

The inequalities now simplify to

\[
\begin{align*}
    b - c & \leq a \leq c - b, \\
    b & \leq c - a, \\
    c & \leq b - a.
\end{align*}
\]

Now, combining the first and third inequality, we obtain \( c \leq b - a \leq b + c - b = c \), so \( c = b - a \) as required.

8. How many sitting orderings can one make for 6 boys and 3 girls, if no two girls are to sit next to each other and they are supposed to sit

(a) at a bar
(b) at a round table.

**Solution:** In the first case, we claim the answer is \(30 \cdot 7! = 151200\).

First, place the boys at the bar. This gives \(6!\) arrangements. Now, there are 7 possible places between boys. Each girl may go in one of these, but may not go in a place that already has another girl in it. Thus, the first girl has 7 places to go, the next 6 and the last one has 5. This gives \(7 \cdot 6 \cdot 5 \cdot 6! = 30 \cdot 7!\) arrangements.

In the second case, we designate one of the girls to be the "head" of the table, in order to break the symmetry. We now have \(6!\) arrangements of the boys around the table, clockwise. Now, there are only 5 places between boys that are free, so we insert the remaining two girls in these places in \(5 \cdot 4\) ways.

This gives \(6! \cdot 5 \cdot 4 = 20 \cdot 6! = 14400\) ways.

9. There are 100 students at a high school who play at least one of the following sports: Tennis, Badminton and Ping-pong. 28 play only tennis, 30 only Badminton and 20 only Ping-pong. 11 students play Badminton and Ping-pong, 9 students play Tennis and Ping-pong and 8 students Tennis and Badminton. How many students play all three sports?

**Solution:**

The information tells us that 22 students play 2 or more sports. Let \( x \) be the number of students who play all three sports. Then, these students are counted in each of the 3 counts of students who play 2 sports. Thus, \(22 = 11 + 9 + 8 - 2x\).

This gives \(x = 3\) students who play all three sports.

10. Given that numbers \( x, y, z \geq 0 \) satisfying \( x^2 + y^2 + z^2 \leq 1 \), what is the probability that \( x + y + z \leq 1 \)?

The part of the ball in the positive octant has volume

\[
\frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}.
\]
The set of points with \( x + y + z \leq 1 \) gives a tetrahedron contained in the ball. Its base is a right triangle of area \( 1/2 \). Thus, the volume of the tetrahedron is \( \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \).

The desired probability is then \( \frac{1}{\pi} \).