

University of Mississippi

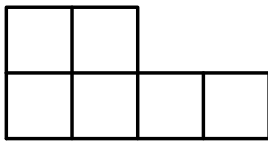
4th Annual High School Mathematics Contest
Team Competition Solutions
October 25, 2008

1. Find z given that

$$\begin{aligned} 2x + 4y + z &= 27 \\ \frac{x}{6} + \frac{y}{3} - z &= 16 \end{aligned}$$

Solution. Multiply the bottom equation by -12 and add the two equations to obtain $13z = -165$ or $z = -165/13$.

2. Prove that an 85×16 grid cannot be tiled (covered without overlap) using tiles of the following shape:



Solution. The area of one tile is 6, so the area of any region which may be tiled using these tiles is a multiple of six. However, $85 \cdot 16 = 2^4 \cdot 5 \cdot 17$ is not divisible by 6. \square

3. Prove that for all real numbers a , b , and c , we have

$$2a^2 + b^2 + c^2 \geq 2a(b + c).$$

Solution. Move everything to the left hand side, distribute the $2a$ and write $2a^2$ as $a^2 + a^2$:

$$a^2 + b^2 - 2ab + a^2 + c^2 - 2ac \geq 0.$$

Now each trinomial may be factored, and we have shown that the original inequality is equivalent to $(a - b)^2 + (a - c)^2 \geq 0$, which is true because squares of real numbers are nonnegative. \square

4. An integer n is called *perfect* if it equals the sum of all its divisors d with $1 \leq d < n$. For example, 28 is perfect because $1 + 2 + 4 + 7 + 14 = 28$. Let a be a positive integer. Prove that if $2^a - 1$ is prime, then $2^{a-1}(2^a - 1)$ is perfect.

Solution. Define $p = 2^a - 1$ and $n = 2^{a-1}(2^a - 1)$. Since p is an odd prime, $2^{a-1}p$ is the prime factorization of n . Therefore, the proper factors of n are $1, 2, 2^2, \dots, 2^{a-1}$, and $p, 2p, 2^2p, \dots, 2^{a-2}p$. Summing these geometric series and substituting $2^a - 1 = p$ gives

$$\sum_{k=0}^{a-1} 2^k + \sum_{k=0}^{a-2} 2^k p = \frac{2^a - 1}{2 - 1} + p \frac{2^a - 1 - 1}{2 - 1} = 2^{a-1}p = n. \quad \square$$

5. Prove that every integer which can be written as the sum of two cubes may also be written in the form $m(m^2 - 3n)$ for integers m and n .

Solution. Write $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)[(a + b)^2 - 3ab]$, and define $m = a + b$ and $n = ab$. \square

6. A cone whose diameter is equal to its height is inscribed in a sphere of radius 1. What is the volume of the cone?

Solution. Let r denote the radius of the cone, C the center of the sphere, D the center of the cone's base and B a point on the edge of the base of the cone. Looking at triangle CDB , we can see that the height of the cone is $1 + \sqrt{1 - r^2}$. Setting the height equal to the diameter, we get $1 + \sqrt{1 - r^2} = 2r \Rightarrow r = 4/5$, so that $V = \pi r^2 h / 3 = \pi(4/5)^2 \cdot (8/5) / 3 = 128\pi/375$.

7. A ship located at point P is known to be 20 nautical miles from point C . Point A is 5 nautical miles west of C , and B is 5 nautical miles east of C . Based on radio signals from stations based at point A and B , we know that the distance AP is 6 nautical miles larger than the distance BP . Find the possible locations of the ship, expressed as a number of nautical miles east/west and a number of nautical miles north/south of C .

Solution. Define a coordinate plane with C at the origin and the x -axis passing through A and B . Let x and y be the coordinates of P . Then we know $x^2 + y^2 = 20^2$, and because $AP - BP = 6$ we know that P is on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci at $(\pm c, 0) = (\pm 5, 0)$ and with $a = 6/2 = 3$. Then we calculate $b = \sqrt{c^2 - a^2} = 4$, so that $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Multiplying this equation by -9 and adding it

to $x^2 + y^2 = 400$ gives $y = \pm \frac{4\sqrt{391}}{5}$ and $x = \frac{12\sqrt{26}}{5}$.

Alternate Solution. Use the coordinate representation in the first solution, and write $\sqrt{y^2 + (x+5)^2} - \sqrt{y^2 + (x-5)^2} = 6$ from the Pythagorean theorem. Squaring and substituting the $x^2 + y^2 = 400$ into this equation, we get

$$\sqrt{425 + 10x} - \sqrt{425 - 10x} = 6,$$

which we may square and rearrange to get

$$407 = \sqrt{425^2 - 100x^2}.$$

This gives $x = \sqrt{425^2 - 407^2} / 10$ which may be simplified using difference of squares to obtain the same answer as above.

8. Consider the following sequence of strings of digits:

1, 2, 4, 13, 31, 112, 224, 1003, ...

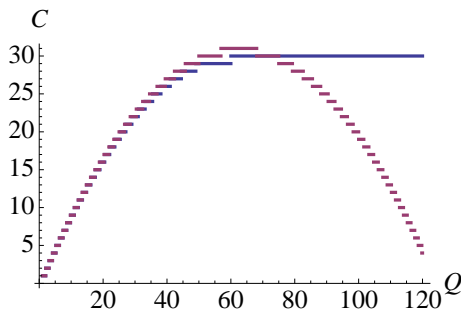
Describe a rule for finding the terms of this sequence, and use it to find the next three terms of the sequence.

Solution. Noticing that the numbers use only the digits 0, 1, 2, 3, and 4, we convert the sequence to base 5, which gives 1, 2, 4, 8, 16, 32, 64, 128, 256, ... Therefore, the next three terms are the base 5 representations of 512, 1024, and 2048, i.e. $\boxed{4022, 13044, \text{ and } 31143}$.

9. Mark is taking a long test, and he knows that if he works at a rate of r questions per minute, his accuracy is $1 - r/5$ (e.g. he will get 80% of the questions that he works at a rate of 1 question per minute correct). There is no partial credit, and he gets no credit for problems he doesn't work completely. With 25 minutes left, he has Q questions remaining. He faces a decision between (a) losing one minute of test time to calculate his optimum rate for the remaining 24 minutes, or (b) continuing at a rate of $Q/25$ questions per minute so that he finishes the test. Determine the values of Q for which strategy (a) will result in a better score than (b).

Solution. For real x , let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x . Working at a rate of r questions

per minute for n minutes will allow Mark to work nr total questions (as long as nr is not more than the number of questions

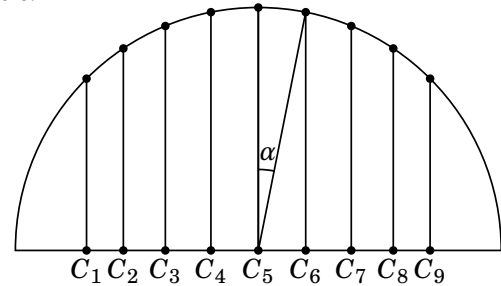


remaining), and therefore he will get $nr(1 - r/5)$ questions correct. So for choice (b), Mark will get $\lfloor Q(1 - Q/125) \rfloor$ questions correct. Now to determine how many questions Mark will get with choice (a), we first optimize the function $C(Q, r) = 24r(1 - r/5)$ for large Q . As a function of r , this is a concave parabola, so its maximum occurs halfway between its two roots $r = 0$ and $r = 5$, i.e. at $r = 5/2$. Therefore, the maximum of $C(Q, r)$ is 30 for $Q \geq 24(5/2) = 60$. For $Q < 60$, it is best to decrease the rate from 2.5 questions per minute to $Q/24$, since he has time to finish the questions, and he increases his accuracy by slowing down. This gives him $\lfloor Q(1 - Q/120) \rfloor$ questions correct for $Q < 60$. Plotting these two functions together and solving $30 = Q(1 - Q/125)$, we see that (a) is better than (b) for $\boxed{Q > 75}$.

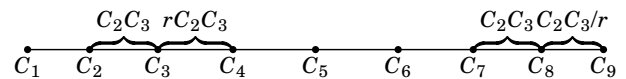
10. To account for an optical illusion which creates an impression of unevenness, the columns of the Parthenon were designed to be slightly closer together near the edges of each facade than in the middle. Consider the following two ways of placing $2n + 1$ points $C_1, C_2, \dots, C_{2n+1}$ on a line, symmetric about the middle point and more closely spaced near the edges than in the middle:

- (i) Choose $\alpha > 0$ and $R > 0$, and equally space $2n + 1$ points with angular separation α on the circumference of a circle of radius R . Then project all the

points onto the diameter perpendicular to the radius drawn to the middle point. We require that $n\alpha \leq 90^\circ$ so that all the points lie on the same semicircle.



- (ii) Ensure that ratios between adjacent column spacings are equal. That is, choose a real number $r > 1$ and place $C_1, C_2, \dots, C_{2n+1}$ so that $rC_k C_{k+1} = C_{k+1} C_{k+2}$ for all $1 \leq k < n$, $C_n C_{n+1} = C_{n+1} C_{n+2}$, and $C_k C_{k+1} = rC_{k+1} C_{k+2}$ for all $n < k < 2n$. (Here $C_i C_j$ denotes the distance from C_i to C_j .)



Prove that (i) and (ii) represent fundamentally different ways of determining column placement. In particular, prove that for all $n \geq 3$, there do not exist r, R , and α so that the points obtained from construction (i) coincide with the points obtained from construction (ii). (Note: the distance $C_1 C_{2n+1}$ between the first and last points is assumed to be fixed.)

Solution. First, from the triangles in the upper figure we see that the distances $C_n C_{n+k}$ in construction (i) are $R \sin(k\alpha)$. Therefore, if for some $n \geq 3$ constructions (i) and (ii) coincide, then we would have

$$\frac{\sin 2\alpha - \sin \alpha}{\sin \alpha - \sin 0} = \frac{\sin 3\alpha - \sin 2\alpha}{\sin 2\alpha - \sin \alpha},$$

by looking at the ratios $C_{n+3} C_{n+2} / C_{n+2} C_{n+1}$ and $C_{n+2} C_{n+1} / C_{n+1} C_n$. Now recall that for all u and v , $\sin u - \sin v = 2 \sin(\frac{u-v}{2}) \cos(\frac{u+v}{2})$. Using this identity to simplify all the differences in the previous equation gives

$$\frac{\cos(3\alpha/2)}{\cos(\alpha/2)} = \frac{\cos(5\alpha/2)}{\cos(3\alpha/2)}.$$

Abbreviating $\beta = 3\alpha/2$, we can rewrite this equation as

$$\cos^2 \beta = \cos(\beta - \alpha) \cos(\beta + \alpha),$$

and we can use the cosine sum-angle formulas: $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$. We get

$$\cos^2 \beta = \cos^2 \beta \cos^2 \alpha - \sin^2 \beta \sin^2 \alpha,$$

and moving everything to the left side and rewriting $1 - \cos^2 \alpha = \sin^2 \alpha$, we get $\sin^2 \alpha (\cos^2 \beta + \sin^2 \beta) = 0$. This is a contradiction since $\cos^2 \beta + \sin^2 \beta = 1$ and $0 < \alpha \leq \pi/6 \Rightarrow \sin^2 \alpha \neq 0$. \square