University of Mississippi

3rd Annual High School Mathematics Contest
Individual Competition
Answer Key and Selected Solutions
October 27, 2007

Answer Key

1. B
2. D
3. A
4. B
5. E
6. B
7. C
8. C
9. C
10. A
11. D
12. B
13. B
14. A
15. A
16. C
17. E
18. D
19. E
20. A
21. E
22. C
23. D
24. C
25. A

Selected Solutions

21. For θ in the first quadrant, the trigonometric expression $\sqrt{1+\sin(2\theta)}$ may be written uniquely in the form $a\cos(b\pi-\theta)$, where a and b are real numbers with $0 \le b \le 1/2$. Find a^2b^2.

(a) 2
(b) 1
(c) 1/2
(d) 1/4
(e) 1/8

Solution. Rewrite 1 as $\cos^2\theta+\sin^2\theta$ and apply the double-angle identity for sine:

$$\begin{aligned}\sqrt{1+\sin 2\theta} &= \sqrt{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}\\ &= \sqrt{(\cos\theta+\sin\theta)^2}\\ &= |\cos\theta+\sin\theta|\\ &= \cos\theta+\sin\theta\\ &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)\\ &= \sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right)\\ &= \sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right).\end{aligned}$$

Dropping the absolute value bars is justified because θ is a first quadrant angle. Then $a=\sqrt{2}$ and $b=1/4$, and $a^2b^2 = \boxed{1/8}$.

22. Suppose that $f(x)$ is a polynomial such that when $f(x)$ is divided by x^2+x+1, the quotient is $x^{101}-x^{99}+6x^{45}$, and the remainder is 0. Let $r(x)$ be the remainder when $f(x)$ is divided by x^3-1. Find $r(2)$.

(a) 6
(b) 30
(c) 42
(d) 49
(e) None of the above

Solution. We are given that

$$\frac{f(x)}{x^2+x+1} = x^{101}-x^{99}+6x^{45},$$

and we notice by difference of cubes that

$$x^3-1=(x-1)(x^2+x+1).$$

Therefore,

$$\begin{aligned}\frac{f(x)}{x^3-1} &= \frac{1}{x-1}\cdot\frac{f(x)}{x^2+x+1}\\ &= \frac{x^{101}-x^{99}+6x^{45}}{x-1}\\ &= \frac{x^{99}(x^2-1)}{x-1}+\frac{6x^{45}}{x-1}\\ &= x^{99}(x+1)+\frac{6x^{45}-6+6}{x-1}\\ &= x^{99}(x+1)+\frac{6x^{45}-6}{x-1}+\frac{6}{x-1}\\ &= x^{99}(x+1)+6(x^{44}+x^{43}+\cdots+1)+\frac{6(x^2+x+1)}{x^3-1},\end{aligned}$$

so $r(x)=6(x^2+x+1)$, and $r(2)=\boxed{42}$.

23. Given that one of the answer choices is correct, which of the following is equal to $\sqrt[5]{28629151^2}$?

(a) 881
(b) 901
(c) 931
(d) 961
(e) 1021

Solution. Notice that a number is a perfect square if and only if its fifth power is a perfect square, since raising its prime factorization to the fifth power doesn't change the parity of any of the exponents. Therefore, the correct answer is $\boxed{961}$, the only answer choice which is a perfect square.

24. For pairs of real numbers (x_1, x_2) with $-1 \le x_2 \le 1$, find the minimum value of the expression

$$(x_1-x_2)^2+\left(10-2x_1-\sqrt{1-x_2^2}\right)^2.$$

(a) $25-2\sqrt{10}$
(b) $25-4\sqrt{6}$
(c) $21-4\sqrt{5}$
(d) $19-6\sqrt{2}$
(e) None of the above

Solution. Write $y_1 = 10-2x_1$, and $y_2=\sqrt{1-x_2^2}$. Then the problem asks us the minimize the distance between a point (x_1, y_1) on the line $y=10-2x$ and a point (x_2, y_2) on a semicircle $y=\sqrt{1-x^2}$. Geometrically, it is clear that this happens when both points are on the line $y=\frac{1}{2}x$ which is perpendicular to the line $y=10-2x$ and passes through the origin. Thus $(x_1, y_1)=(4,2)$, and the distance in question is

$\sqrt{4^2+2^2}-1=2\sqrt{5}-1$. Therefore minimum value of the expression is $(2\sqrt{5}-1)^2 = \boxed{21-4\sqrt{5}}$.

25. There exist two real solutions x_1 and x_2 of the equation

$$\left(\sqrt{3}-x\right)\left(1+\sqrt{x^2+1}\right)=x\left(1+x\sqrt{3}\right).$$

For any real t, let $\operatorname{Arctan} t$ be the number θ such that $-\pi/2<\theta<\pi/2$ and $\tan\theta=t$. To the nearest thousandth, find $\operatorname{Arctan} x_1+\operatorname{Arctan} x_2$.

(a) −0.698

(b) 0.573

(c) −0.106

(d) 0.391

(e) None of the above

Solution. Write $x=\tan t$ for $-\pi/2<t<\pi/2$ (since tangent maps $(-\pi/2,\pi/2)$ onto $(-\infty,\infty)$) and write $\sqrt{3}=\tan\frac{\pi}{3}$. Rearranging some factors, we have

$$\frac{\tan\frac{\pi}{3}-\tan t}{1+\tan\frac{\pi}{3}\tan t}=\frac{\tan t}{1+\sqrt{\tan^2 t+1}}.$$

Using $\tan^2 t+1=\sec^2 t$, we simplify the right-hand side to $\tan t/(1+|\sec t|)$, which equals $\tan t/(1+\sec t)$ because $-\pi/2<t<\pi/2$. Multiplying numerator and denominator by $\cos t$, we recognize the identity $\sin t/(1+\cos t)=\tan(t/2)$. On the left-hand side, we recognize the tangent difference-angle identity, so we have

$$\tan(\pi/3-t)=\tan(t/2).$$

By looking at the graph of tangent, we can see that the tangents of two angles are equal if and only if they differ by an integer multiple of π. Therefore, we may rewrite the equation as $\pi/3-t-t/2=k\pi$, where k is any integer. Solving for t, we obtain $t=2\pi/9-2\pi k/3$, and the values of k for which t is in the range $-\pi/2<t<\pi/2$ are $k=0$ and $k=1$, for which $t=2\pi/9$ and $t=-4\pi/9$ respectively. Therefore, $\operatorname{Arctan} x_1+\operatorname{Arctan} x_2=2\pi/9-4\pi/9=-2\pi/9$, and a bit of long division reveals that this corresponds to answer choice (a), $\boxed{-0.698}$.