University of Mississippi

4^{th} Annual High School Mathematics Contest
Individual Competition
October 25, 2008

1. Two solutions of $(t+2)^2 + 12t = 669$ are $t = 19$ and $t = -35$. Which of the following is also a solution of this equation?

(a) 22
(b) 23
(c) −14
(d) −31
(e) None of the above

2. A Latin square is a square table of symbols in which each symbol appears exactly once in every row and column. Find the symbol that belongs in position indicated by the question mark in the following partial Latin square:

☺		❐	☎
❐	☎	?	
✰			
	☺		❐

(a) ☺
(b) ✰
(c) ☎
(d) ❐
(e) Cannot be determined

3. Lagrange's four-square theorem states that every positive integer may be written as the sum of four perfect squares. For example, $38 = 6^2 + 1^2 + 1^2 + 0^2$. Write 88 in the form $a^2 + b^2 + c^2 + d^2$ where a, b, c, and d are integers, and calculate $|a| + |b| + |c| + |d|$.

(a) 16
(b) 18
(c) 20
(d) 24
(e) None of the above

4. How many prime factors does 391 have?

(a) 1
(b) 2
(c) 3
(d) 4
(e) More than 4

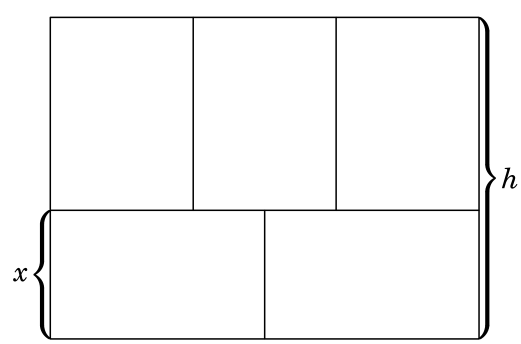
5. In a certain country, the four states A, B, C, and D have populations of 11, 33, 44, and 77, respectively. Presidential elections in this country are based on an electoral college system, and the four states have 1, 3, 4, and 7 electoral college votes, respectively. The candidate who receives the most individual votes in each state gets all of that state's electoral college votes, and the candidate who gets the most electoral college votes wins. In an election between two candidates, what is the minimum number of individual votes that the winning candidate could receive?

(a) 45
(b) 54
(c) 56
(d) 62
(e) None of the above

6. A square's side length is a whole number of centimeters, its diagonal is shorter than 20 centimeters, and its area is A. How many possible values are there for A?

(a) 9
(b) 11
(c) 12
(d) 14
(e) 16

7. Josephine wants to cut a rectangular cake into 5 pieces of equal area as shown. Find the ratio x/h.



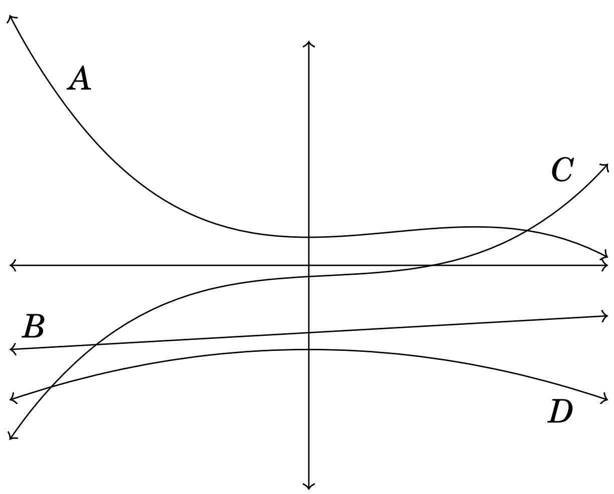
(a) 1/5
(b) 3/7
(c) 1/3
(d) Cannot be determined
(e) None of the above

8. Chester is asked to find every real number which is equal to its cube. His solution is shown below. In which step did he make an error?

$$x^3 = x \quad (1)$$
$$x^2 = 1 \quad (2)$$
$$x^2 - 1 = 0 \quad (3)$$
$$(x+1)(x-1) = 0 \quad (4)$$
$$x+1=0 \quad \text{or} \quad x-1=0 \implies \boxed{x \in \{-1, 1\}} \quad (5)$$

(a) Going from (1) to (2)
(b) Going from (2) to (3)
(c) Going from (3) to (4)
(d) Going from (4) to (5)
(e) No error was made.

9. The graphs of four polynomial functions f, g, h, and $f+g+h$ are shown below. Determine which of the graphs represents the function $f+g+h$.



(a) A
(b) B
(c) C
(d) D
(e) Cannot be determined.

10. Which of the following statements are true?

I. An acute triangle can be isosceles.
II. An obtuse triangle can be isosceles.
III. A right triangle can be isosceles.

(a) I and II
(b) I only
(c) II only
(d) I, II, and III
(e) None of the statements are true.

11. Which of the following is equal to $4 \times 0.\overline{63}$?

(a) $2.\overline{55}$
(b) $2.\overline{54}$
(c) $2.\overline{53}$
(d) $2.\overline{52}$
(e) $2.\overline{5}$

12. Which of the following numbers is closest on the real number line to the point which is halfway from 10^{-13} to 10^{-7}?

(a) 10^{-7}
(b) 10^{-8}
(c) 10^{-9}
(d) 10^{-10}
(e) 10^{-13}

13. To the nearest whole percent, 89% of the people that work with Mitchell keep plants in their office. What is the sum of the digits of the minimum possible number of people who work with Mitchell?

(a) 9
(b) 7
(c) 6
(d) 3
(e) None of the above

14. For how many of the following numbers x is it true that $\sqrt[4]{x^2} = \sqrt{x}$?

$$4, \quad \frac{13}{2}, \quad -\sqrt{2}, \quad \pi, \quad -3$$

(a) 0
(b) 2
(c) 3
(d) 4
(e) None of the above

15. What is the maximum number of nonzero entries in a 2×2 matrix whose square is the zero matrix?

(a) 1
(b) 2
(c) 3
(d) 4
(e) None of the above

16. Evaluate $16^5 - 12 \cdot 16^4 - 60 \cdot 16^3 - 60 \cdot 16^2 - 54 \cdot 16 + 10$.

(a) -16
(b) 90
(c) 170
(d) 232
(e) None of the above

17. Let $r = \sqrt{111} - \sqrt{110} + \sqrt{10000^{-1}}$. Which of the following is true?

(a) $5 < 100r < 5.5$
(b) $5.5 < 100r < 6$
(c) $6 < 100r < 6.25$
(d) $6.25 < 100r < 6.5$
(e) None of the above

18. Sam's grandfather was born in 1918, and Sam was born in 1986. Therefore, in 2004, Sam's grandfather was 86 years old, having been born in '18, while Sam was 18 years old, having been born in '86. Suppose that the birth year of person A is chosen uniformly at random between the years 1901 and 1950 inclusive, while the birth year of person B is chosen uniformly at random between 1951 and 1999 inclusive. What is the probability that there will be a year in which person A's age is equal to the number of years between 1900 and the birth year of person B, and vice versa?

(a) 1/5
(b) 1/4
(c) 1/3
(d) 1/2
(e) None of the above

19. Eight numbers are chosen uniformly at random (independently of one another) in the interval [0,1]. What is the probability that the second largest of the numbers is less than 0.9?

(a) $(0.9)^7$
(b) $(0.9)^7 + (0.9)^8$
(c) $0.1 \cdot (0.9)^8$
(d) $1.7 \cdot (0.9)^7$
(e) None of the above

20. What is the least integer n such that there is exactly one perfect square among the numbers n, $n+1$, …, $n+1000$?

(a) 562501
(b) 250001
(c) 62501
(d) 22501
(e) None of the above

21. Find the greatest integer which is not greater than

$$\log_3\left(3^{97} + 3^{96} + 3^{95} + \cdots + 3^{33}\right).$$

(a) 101
(b) 99
(c) 98
(d) 97
(e) None of the above

22. For each $x \in [-1,1]$, let the function $\cos^{-1}(x)$ denote the usual inverse of the cosine function. That is, $\cos^{-1}(x)$ is the unique $\theta \in [0,\pi]$ with $\cos\theta = x$. Solve

$$\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}(x) = \cos^{-1}\left(-\frac{63}{65}\right).$$

(a) $\frac{4}{5}$
(b) $\frac{7}{24}$
(c) $-\frac{8}{17}$
(d) $-\frac{7}{24}$
(e) None of the above

23. A list of 177 positive numbers has a unique mode. The number 3 appears exactly 45 times in the list, the number 2 appears exactly 64 times, and the sum of the 177 numbers is 885. Find the largest possible sum of the mean, the median, and the mode of the numbers.

(a) 9
(b) 11
(c) 17
(d) 19
(e) None of the above

24. Define $n(a,b,c)$ to be the number of real solutions x of the equation

$$\sqrt{x-a} + \sqrt{x-b} = \sqrt{c}.$$

Find $n(1,2,3) + n(4,4,0) + n(4,4,4) + n(25,15,5)$.

(a) 1
(b) 2
(c) 3
(d) 4
(e) None of the above

25. Find the remainder when $x^{48} - 9x^{47} + 20x^{46} + x$ is divided by $x^2 - 4x - 5$.

(a) 29
(b) $-4x + 25$
(c) $-4x + 29$
(d) $x + 29$
(e) None of the above

26. Find the least value of a so that the graph of the function $g(x) = -\sqrt{a^2 - x^2 - 12x - 36} + 9$ intersects the graph of $h(x) = \sqrt{4a^2 - x^2} + 1$.

(a) 10/3
(b) 5
(c) $5\sqrt{2}$
(d) $5\sqrt{3}$
(e) None of the above

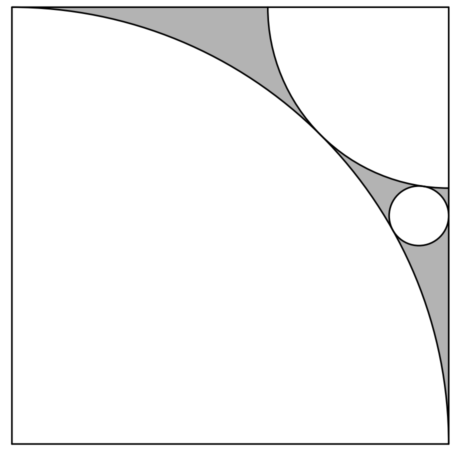
27. There is an (nondegenerate) equilateral triangle $\triangle ABC$ having all three vertices on the right branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with $A = (a, 0)$. Find the length of an altitude of this equilateral triangle.

(a) $\dfrac{a^2b + 3ab^2}{a^2 - 3b^2}$
(b) $\dfrac{a^3 + 3a^2b}{a^2 - 3b^2}$
(c) $\dfrac{a^2b + 3ab^2}{a^2 + 3b^2}$
(d) $\dfrac{a^3 + 3a^2b}{a^2 + 3b^2}$
(e) None of the above

28. Jamye won a large number of 2% off coupons for purchases at Blue Bottom Jeans. She is allowed to apply as many coupons as she wishes on each purchase, and the discounts are applied successively. After all the discounts are applied, a tax of 8.5% is added. What is the smallest number of coupons she must use in order for the total cost of a pair of blue jeans (after tax) to be less than the original price of the jeans? (Note: It might be useful that $1.085^{-1} = 0.92165...$).

(a) 2
(b) 3
(c) 4
(d) 5
(e) None of the above

29. The two quarter-circles shown below are externally tangent with centers at opposite vertices of a unit square, and the larger quarter-circle passes through two vertices of the square. A small circle is drawn externally tangent to each of the two quarter-circles and to the side of the square. What is the radius of the small circle?



(a) $\dfrac{5\sqrt{2} - 7}{2}$
(b) $\dfrac{2\sqrt{2} - 1}{32}$
(c) $\dfrac{9 - 4\sqrt{2}}{49}$
(d) $\dfrac{11 - 6\sqrt{2}}{128}$
(e) None of the above

30. Any given toss of a certain weighted coin has a probability p of resulting in heads and $1 - p$ of resulting in tails. The probability that an even number of heads will result in 99 tosses of the coin is 3/4. Find p.

(a) $1/2 - 2^{-100/99}$
(b) $1/2 + 2^{-99}$
(c) $1/2 - 2^{-99}$
(d) $1 - 2^{-1/99}$
(e) None of the above