University of Mississippi
12^{th} Annual High School Mathematics Contest
Team Competition. February 11, 2017.

1. If $a_1, a_2, ..., a_{2107}$ are consecutive integers such that $a_1 < a_2 < a_3 < \cdots < a_{2017}$, and

$$a_1 - a_2 + a_3 - a_4 + \cdots - a_{2016} + a_{2017} = 2017,$$

show that a_{2017} is a perfect square.

2. Each of the numbers 1, 2, 3, 4, 5 and 6 are painted on the faces of a cube, one number per face. (The result is similar to a die, but may have a different ordering of the numbers.) The cube is placed on a table. From each of three chairs, a person can see the top and two other faces of the cube. The sums of the numbers on the visible faces, from each of these three chairs, are 9, 14 and 15. What is the number on the bottom face?

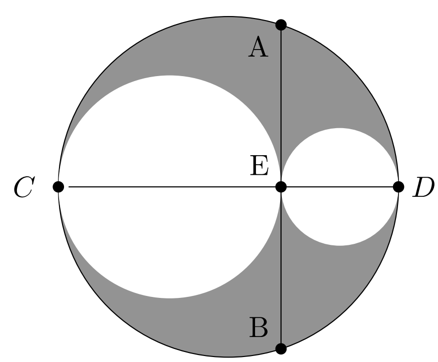
3. What is the smallest number of consecutive positive integers that add up to 1000? (You need to use at least two numbers.) What are these integers?

4. Find all possible quadruplets of positive numbers a, b, c, d so that

$$abcd = 1 \quad \text{and}$$
$$ab + bc + cd + da + a + b + c + d = 8.$$

5. Alice, Bob and Charlie play a game that involves throwing a die. Alice throws first, then Bob and then Charlie. Alice wins if she throws 1, 2 or 3. Bob wins if he throws a 4 or a 5. Charlie wins if he throws a 6. They repeat like this until someone wins. What is the probability that Charlie wins?

6. In the diagram, the line AB is tangent to the unshaded circles, and has length 10 cm. The centers of all 3 circles lie on the line CD, which has length 11 cm. Find the area of the shaded region.



7. If x, y are real numbers so that $x^2 + 3xy + y^2 = 60$, what is the maximum possible value of xy?

8. Show that there are no positive integer solutions to the equation

$$x^x + 2y^y = z^z.$$

9. Two players play the following game. They start with a pile of 13 stones. Each player, in turn, removes 1, 2 or 3 stones from the pile. They take turns until there are no more stones in the pile. The last person who removes a stone *loses* the game.

(i) Which player has a winning strategy, the first or the second?

(ii) Suppose they start with n stones instead. For what values of n does the first player have a winning strategy?

10. What is the smallest positive integer k so that $k!$ is a multiple of 2016?

Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$.