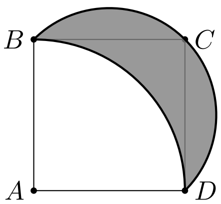
The University of Mississippi

13th Annual High School Mathematics Contest

Individual Competition (February 10, 2018)

1. The square below has side length 1. Find the area of the shaded region bounded by arcs of two circles. The sides BC and CD are tangent to the shorter arc.



(a) 1
(b) $\pi/4$
(c) $3/4$
(d) $\frac{1}{2}$
(e) None of the above

2. Ann and Bob are walking towards each other. Ann is walking at a constant speed of 2 miles per hour. Bob is walking at a constant speed of 3 miles per hour. At the moment when the distance between Ann and Bob is 1 mile, Ann's dog starts running towards Bob, until it reaches him, then turns back and starts running towards Ann until it reaches her, then again towards Bob, and so on, until Ann and Bob meet. If the dog runs at a constant speed of 10 miles per hour, what is the total distance traveled by the dog?

(a) 1 miles
(b) 2 miles
(c) 3 miles
(d) 5 miles
(e) None of the above

3. What is the expression

$$\frac{\sqrt{30+10\sqrt{5}}-\sqrt{14+6\sqrt{5}}}{10}$$

equal to?

(a) 1/10
(b) 1/5
(c) 3/10
(d) 2/5
(e) None of the above

4. Simplify $\left(\frac{1}{2}\right)^{\log_4 \frac{1}{x^2}}$

(a) $|x|$
(b) x
(c) x^{-1}
(d) x^2
(e) None of the above

5. If the solution to the equation $3^{x+3} = 4^x$ is $x = 3\log_b 3$, what is b?

(a) 3/2
(b) 4/3
(c) 5/3
(d) 9/4
(e) None of the above

6. If f and g are functions of three real variables satisfying $f(x,y,z) = (y,z,x)$ and $g(x,y,z) = (y,x,z)$, for every real x,y,z, what is $f \circ g \circ f^{-1}(x,y,z)$? Here, $f \circ g$ denotes the usual composition of functions defined by $f \circ g(x,y,z) = f(g(x,y,z))$ and f^{-1} is the inverse function of f satisfying $f^{-1} \circ f(x,y,z) = (x,y,z)$, for all real x, y and z.

(a) (x,y,z)
(b) (x,z,y)
(c) (y,x,z)
(d) (z,y,x)
(e) None of the above

7. When the expression $(3x-4)^9$ is expanded in powers of x, what is the sum of the coefficients?

(a) -2
(b) -1
(c) 0
(d) 1
(e) 2

8. We draw cards, one at a time, at random and successively from an ordinary deck of 52 cards, replacing the drawn card every time. What is the probability that an ace appears before a face card?

(a) 1/3
(b) 1/4
(c) 1/8
(d) 1/12
(e) 1/13

9. Suppose a sequence a_n is defined for $n = 1, 2, 3, \ldots$ as $a_1 = 1$, $a_{2n} = 2a_{2n-1}$, and $a_{2n+1} = a_{2n} + 1$. Which one of the following numbers is closest to $\sqrt[2018]{a_{2018}}$?

(a) 1.4
(b) 1.5
(c) 1.6
(d) 1.8
(e) 2

10. How many zeros does $53! = 1 \cdot 2 \cdots 53$ end with?

(a) 10
(b) 12
(c) 14
(d) 15
(e) 20

11. In a triangle ABC, $\cos A + \cos B = \frac{\sqrt{2}}{2}$ and $\sin A - \sin B = \frac{\sqrt{2}}{2}$. What is the inner angle at vertex C?

(a) $30°$
(b) $45°$
(c) $60°$
(d) $90°$
(e) $120°$

12. Consider $x^2 + px + q = 0$, where p and q are positive numbers. If the solutions of this equation differ by 1, then p equals

(a) $\sqrt{4q+1}$
(b) $q - 1$
(c) $-\sqrt{4q+1}$
(d) $q + 1$
(e) $\sqrt{4q-11}$

13. Assume that, for a certain school, it is true that

1. Some students do not wear brown shoes, and
2. All fraternity members wear brown shoes.

Which of the following must be true?

(a) Some students are fraternity members
(b) Some fraternity members are not students
(c) Some students are not fraternity members
(d) No fraternity member is a student
(e) No student is a fraternity member

14. Given the linear fractional transformation of x into $f(x) = \frac{2x-1}{x+1}$. Define the usual compositions of functions $f^{n+1} = f(f^n(x))$ for $n = 1, 2, \ldots$. It can be shown that $f^{35} = f^5$; it follows that $f^{28}(x)$ is:

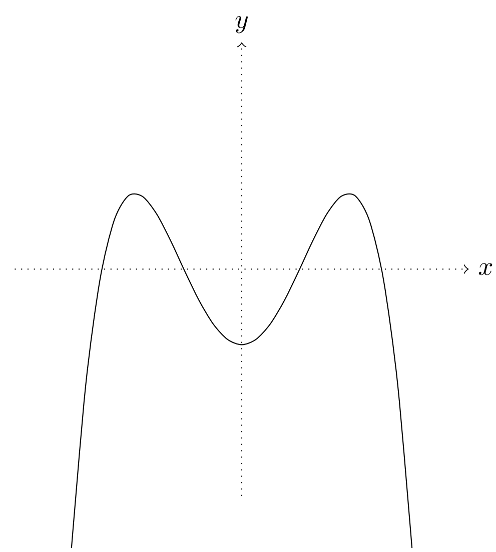
(a) x
(b) $\frac{1}{x}$
(c) $\frac{x-1}{x}$
(d) $\frac{1}{1-x}$
(e) none of the above

15. A drawer contains red socks and black socks. When two socks are drawn at random, the probability that both are red is 1/2. Furthermore, the number of black socks is even. What could the number of socks be?

(a) 4
(b) 8
(c) 13
(d) 19
(e) 21

16. The following is the graph of some function

$$f(x) = ax^4 + bx^2 + c.$$



What are the signs of the constants a, b and c?

(a) $a > 0,\ b > 0,\ c > 0$
(b) $a > 0,\ b < 0,\ c > 0$
(c) $a > 0,\ b < 0,\ c < 0$
(d) $a > 0,\ b > 0,\ c < 0$
(e) $a < 0,\ b > 0,\ c < 0$

17. What is the last digit of 3^{2018}?

(a) 1
(b) 3
(c) 5
(d) 7
(e) 9

18. The cubic equation $x^3 - x^2 + 2x - 3 = 0$ has three roots, x_1, x_2 and x_3. What is $x_1 + x_2 + x_3$?

(a) -1
(b) 1
(c) 2
(d) 3
(e) None of the above

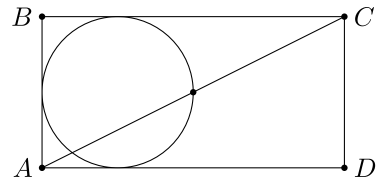
19. A sphere S_1 is inscribed in a cube C_1; a cube C_2 is inscribed in the sphere S_1; a sphere S_2 is inscribed in the cube C_2; and so on. What is the ratio $Area(C_1)/Area(C_{10})$ of the surface areas of C_1 and C_{10}?

(a) 3^9
(b) 3^8
(c) $3^{9/2}$
(d) 81
(e) None of the above

20. An equilateral triangle in inscribed in a circle. If a point is chosen at random inside the circle, what is the probability that the point is outside of the triangle?

(a) $1 - \frac{\sqrt{3}}{\pi}$
(b) $1 - \frac{3\sqrt{3}}{2\pi}$
(c) $1 - \frac{3\sqrt{3}}{4\pi}$
(d) 1/2
(e) None of the above

21. A circle of radius r is tangent to the sides AB, AD and CD of a rectangle $ABCD$ and passes through the midpoint of the diagonal AC. The area of the rectangle, in terms of r, is



(a) $4r^2$
(b) $6r^2$
(c) $8r^2$
(d) $12r^2$
(e) $20r^2$

22. Chris is walking down an airport escalator with a constant speed. When the escalator is at rest, it takes him 30 sec to come down the escalator, from the upper floor to the lower floor. When the escalator is moving, it takes him 20 sec to come down the escalator. How long would take Chris to come down the moving escalator, if he was just standing on it?

(a) 35 sec
(b) 40 sec
(c) 50 sec
(d) 60 sec
(e) 80 sec